Average exceptional Lie and Coxeter group hierarchies with special reference to the standard model of high energy particle physics

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Abstract

The notions of the order of a symmetry group may be extended to that of an average, non-integer order. Building on this extension, it can be shown that the five classical exceptional Lie symmetry groups could be extended to a hierarchy, the total sum of which is four times 

\[ a_0 = 137 + k_0 \]

of the electromagnetic field. Subsequently it can be shown that all known and conjectured physical fields may be derived by \( E \)-infinity transfinite scaling transformation. Consequently \( E_8 \) exceptional Lie symmetry groups manifold, the SL(2,7) holographic modular curve boundary \( \Gamma(7) \), Einstein–Kaluzu gravity \( R^{(n=4)} \) and \( R^{(n=5)} \) as well as the electromagnetic field are all topological transformations of each other. It is largely a matter of mathematical taste to choose \( E_8 \) or the electromagnetic field associated with \( a_0 \) as derived or as fundamental. However since \( E_8 \) has been extensively studied by the founding father of group theory and has recently been mapped completely, it seems beneficial to discuss at least high energy physics starting from the largest of the exceptional groups.

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1. Introduction

From the outset, the exceptional Lie groups have played a significant role in developing \( E \)-infinity theory [1,2]. In particular the fundamental scaling relation between \( \text{Dim} E_8 E_8 \simeq 496 \) and \( a_0 \simeq 137 \) of electromagnetism

\[ a_0 = (496 - k^2)/(3 + \phi) = 137 + k_0 = 137.082039325 \]

where \( \phi = (\sqrt{5} - 1)/2, k = \phi^3(1 - \phi^3) \) and \( k_0 = \phi^5(1 - \phi^5) \) as well as the mass of the expectation \( \pi \) and \( K \) mesons

\( \langle \pi \rangle = a_0 \text{ Mev} \)

and

\( \langle K \rangle = \text{Dim} E_8 E_8 \text{ Mev} \)

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were found to be more than a telling indication of the profundity of the $E_8$ exceptional Lie group and the $\phi$ scaling of $E$-infinity theory [3,4].

In a large number of recent papers by Ji-Huan He in China, L. Marek-Crnjac in Slovenia and the present author, the role of the exceptional and the Coxeter symmetry groups in connection with high energy physics were discussed and analysed extensively [4,5]. The basic idea upon which our work in $E$-infinity theory is based is a logical extension of notions and functions based entirely on integers to include rational and irrational numbers. Thus following the extension of the factorial function $n!$ to the gamma function as well as the topological dimension to a Hausdorff fractal dimension, we extend the meaning of the order of a symmetry group to that of an average symmetry with non-integer numbers as its order [6–8]. In fact N.H. Ahmed in Newcastle, England suggested that the success of $E$-infinity theory lies in using a golden mean based number system [9]. However this is only part of the story. The second part lies in the particular nice result of this theory was a recent proof of quarks confinement [10,11].

In the present paper we aim to further explore the role played by $E_8$ in high energy physics and transfinite scaling transformations connected to it.

2. The exceptional Lie groups hierarchies

Various important exceptional Lie group hierarchies were suggested recently with remarkable properties reminiscent of the Freudental magic square. In particular all these hierarchies sum up to (4) $(2_0)$ as shown in earlier publications [5]:

$$\sum_{j=1}^{j=n} |E_i| = 548 = (4)(137).$$

Taking the exact transfinite value of $|E_i|$ the result is an irrational number

$$\sum_{j=1}^{j=n} |E_i|_k = 548 + 4k_0 = 548.328172 = (4)(2_0)$$

where $k_0 = \phi^5(1 - \phi^5)$ and $\phi$ is the golden mean. To be more explicit let us consider the following hierarchy [5]

$$|E_8| + |E_7| + |E_6| + |SO(10)| + |SU(5)| = |E_8| + |E_7| + |E_6| + |E_5| + |E_4| = 248 + 133 + 78 + 45 + 24 = 528.$$  

This is exactly equal to the number of Brane states in E. Witten’s famous $D = 11$ model [4,5]

$$N(Brane) = \binom{11}{1} + \binom{11}{2} + \binom{11}{5} = 11 + 55 + 462.$$  

It is important to see that the first term corresponds to string-like one-dimensional objects and is equal to the spacetime dimensionality of $M$ theory. The second term on the other hand corresponds to the number of the independent components of the Riemannian tensor for a Kaluza–Klein space $R^{(5)} = 50$ plus the $D = 5$

$$R^{(5)} + D^{(5)} = \frac{(5^2(5^2 - 1)}{12} + 5 = 50 + 5 = 55.$$ 

Finally the following interpretation of the third term is particularly interesting. Recalling the intrinsic dimensionality of the $E_8$ manifold $D = 57$, then we may write

$$\binom{11}{5} = (57)(D^{(8)}) + D^{(6)} = 456 + 6 = 462$$

where $D^{(8)} = 8$ is the super space $4 + 4 = 8$ and $D^{(6)} = 6$ is the compactification orbifold. In addition we know that 528 is the number of isometries in a maximally symmetric manifold with $D = 4$ when lifted to $D = 8$. That means for $D = 32$ we have [3–5]

$$N_{(32)} = n(n + 1)/2 = 32(32 + 1)/2 = (16)(33) = 528.$$  

The corresponding Killing vector field for $D = 11$ is

$$N_{(11)} = 11(11 + 1)/2 = 66.$$
This is exactly equal to $N$-string plus $N$-membranes
\[
\binom{11}{1} + \binom{11}{2} = 11 + 55 = N_{k}^{(11)} = 66
\]
which is the number of particles in MSSM with two Higgs doublets. We not on passing that the exact integer number is 69 which is found from
\[
|\text{SO}(10)| + |\text{SU}(5)| = 45 + 24 = 69
\]
while the 66 is found from the $F_{4}$ and $G_{2}$ exceptional Lie groups [5]
\[
|F_{4}| + |G_{2}| = 52 + 14 = 66
\]
as we will shortly see.

Returning to our hierarchy let us add the standard model SM and the Higgs field $H$ to our 528 and find
\[
\sum_{i=2}^{n} |E_{i}| = 528 + |\text{SM}| + |H| = 528 + |E_{3}| + |E_{2}| = 528 + 12 + 8 = 548 = (4)(137)
\]
extactly as anticipated. Now we could change some of the members of the hierarchy without changing the invariant total dimensions by involving $F_{4}$ and $G_{2}$ exceptional group and removing the unification groups $\text{SO}(10)$ and $\text{SU}(5)$. In addition we replace $|H| = 8$ by $D = 11$. Proceeding this way one finds
\[
\sum_{i=2}^{n} E_{i} = |E_{8}| + |E_{7}| + |E_{6}| + |F_{4}| + |G_{2}| + |\text{SM}| + D^{(11)} = 248 + 133 + 78 + 52 + 14 + 12 + 11 = 459 + 52 + 14 + 12 + 11 = 459 + 66 + 23 = 548.
\]
Thus the number of elementary particles of the standard model is given in this case by
\[
N(\text{SM}) = |F_{4}| + |G_{2}| = 52 + 14 = 66.
\]
Next we analyze the dimension 548 and by putting it under our number theoretical microscope, however never without physical interpretation. We know that $|E_{8}|E_{8}| = 496$ is used in superstrings and the first symmetry breaking leads to a shadow world branching out, so we are left with $|E_{8}| = 496/2 = 248$. Now the bosonic string space is 26-dimensional and if added to 248 then we have 248 + 26 = 274 = $2\tilde{a}_{0}$. Thus recombining them like $|E_{8}|E_{8}|$ one finds $4\tilde{a}_{0} = 548$. In other words $|E_{8}|E_{8}|$ is found from 548 by subtracting $(2)(26) = |F_{4}| = 52$. Said differently we have
\[
|E_{8}|E_{8}| = 4\tilde{a}_{0} - |F_{4}| = \sum_{2}^{8} |E_{i}| - 52 = 496.
\]
In this connection we recall the amazing relation between 496 and the mass of the average $K$ meson, namely 496 Mev. This, as well as the average $\eta$ particle 548 Mev was explained elsewhere and all as consequences from the natural lack of a natural scale in a fractal-like geometry for spacetime as is the case with $E$-infinity Cantorian spacetime. In fact a far more surprising observation in this connection is the following.

An important scaling exponent used in finding the mass spectrum of quantum particles as well as in cancellation of anomalies is $\lambda = 2\tilde{a}_{0} - 1 = 273$ but then again, this is the absolute zero temperature when using appropriate units. Of course we are moving here on thin ice and could be subjected to Bethe standard joke if we say more than what we have said. Nevertheless, we have moved rather far since Bethe and although we may be confined to a conventional main stream nutshell, we can call ourselves kings of infinite dimensional spaces. Thus we cannot ignore the fact that at $-273^\circ$ an ideal gas volume becomes zero and consequently analogous to matter disappearing into a black hole at least theoretically and barring a phase transition from gas to say a fluid. The situation is far too important to ignore or to be deterred from thinking seriously about just because of an outdated joke although it was at the time (almost 80 years ago) justified based on that times standard of theoretical knowledge. Let us not forget that merely fusing time with space as in Einstein’s theory was quite a revolution. Could anyone imagine what the time at the world would have commented on the Brane world theories in 11 dimensions? To motivate a reappraisal of the situation we may mention the recent report on a ‘test tube universe’. In this study [12] the equations of super fluid are used to simulate elementary particles and cosmic phenomena, such as black holes and subject it to actual experiment in a test tube at just above 273 degrees below the freezing point of water. In this connection let me recall what Nobel laureate Gerard ‘t Hooft wrote in his acclaimed popular book ‘In search of the ultimate building block’ [13]:

“The physics of elementary particles has nothing to do with the physics of low temperature, but the mathematics is very, very similar... what a nice feature of theoretical physics! Totally different worlds can be compared with each other, just because they happen to obey similar mathematical equations.”
In view of that the author would like to view $E$-infinity theory as the general theoretical frame work of not only analogies between different phenomena, but also the analogies between analogies.

3. Irrational scaling transformation of $\sum |E_i| = 528 + k_0$ to the different physical fields

Let us follow Fig. 1 and start from $\sum |E_i|_c$ by scaling it using $\lambda = \varphi$. This gives us

$$\left( \sum_i |E_i|_c \right) (\varphi) = (548 + 4k_0) \left( \sqrt{5} - 1 \right) / 2 = 338.8854383 = 336 + 16k = |\text{SL}(2,7)| + 16k = T_c(7).$$

Thus our sum invariant dimension produced the compactified holographic boundary, Fig. 3, of $E$-infinity using simple scaling transformation [1–3]. Next we repeat the scaling using the universal fluctuation $\varphi^3$ and find

$$\left( \sum_i |E_i| \right) (\varphi^3) = 129.4427192 = 128 + 8k = z_{\text{ew}} + 8k$$

where $z_{\text{ew}}$ is the theoretical value of the electroweak inverse coupling found from $z_0 = 137$ by two-Adic expansion

$$\|z_0\|_2 = \|2^{8^{-1}} + 2^{4^{-1}} + 2^{1^{-1}}\|_2 = \|128 + 8 + 1\|_2 = \|137\|_2 = 1.$$  

A second scaling gives us the dimension of the Narain lattice of super strings

$$\left( \sum_i |E_i| \right) (\varphi^4) = 80.$$

Fig. 1. The exceptional Lie group hierarchy. Note that there are numerous ways to construct such a hierarchy. All of them will include $E_8$, $E_7$ and $E_6$. However the total dimension must remain invariant and equal to $4$ multiplied with the inverse of the electromagnetic fine structure constant $z_0$ and thus equal to $548 + 4k_0 = 548.3281572$. 

To obtain the bulk it is extremely easy to see that
\[
\left( \sum |E_i| \right) \left( \frac{3 + \phi}{4} \right) = \text{Dim} E_8 E_8
\]
while the electromagnetic inverse fine structure constant is given simply as
\[
\sum |E_i| / D^{(4)} = (548 + 4k_0)/4 = 137 + k_0 = \tilde{a}_0.
\]
As in Sir Paul McCartney's and friends old song, all what $E_8 E_8$ needs to do is to “twist and shout” when $E$-infinity space is squeezed in the expectation value of the Hausdorff dimension $\langle d_e \rangle = 4 + \phi^3$ and the topological dimension $n_T = 4$ of spacetime. To unify it all with Einstein’s gravity in one obvious equation, one finds for the Riemannian tensor in $D = 4$ that
\[
R^{(4)} = |E_8 E_8| - [\tilde{a}_0 + I_e(7)] = (496 - k2) - [(137 + k_0) + (336 + 16k)] = 20
\]
in full agreement with
\[ R^{(4)} = n^2(n^2 - 1)/2 = 4^2(4^2 - 1)/2 = 20. \]

Finally Newton’s fine structure constant may be found in a similar way explained elsewhere [14].

### 4. Coxeter groups and Ji-Huan He’s polytopes

It is a proven theorem that any simplicial Coxeter group can be represented as a geometrical reflection group. Spherical and Euclidean irreducible diagrams are given in Table 1. This established clearly the connection of the Coxeter

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<thead>
<tr>
<th>Spherical Diagrams</th>
<th>Euclidean Diagrams</th>
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<tbody>
<tr>
<td>$A_n$</td>
<td>$\bar{A}_n$</td>
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<tr>
<td>$B_n$</td>
<td>$\bar{B}_n$</td>
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<tr>
<td>$D_n$</td>
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<td>$I_2(p)$</td>
<td>$\bar{D}_n$</td>
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Table 1
Irreducible spherical and Euclidean diagrams
group to our four exceptional groups $E_6$, $E_7$, $E_8$, $F_4$ as well as $H_4$ corresponding to $G_2$. Furthermore, the regular Coxeter polytopes are well known examples of Coxeter groups. For instance the four-dimensional polytopes called 24-cell, 120-cell and 600-cell were used in crisp and fuzzy form in connection with $E$-infinity theory by L. Marek-Crnjac and the author [6,7] Interestingly the 24-cell is self dual and together they relate to $E_8$ which is quite obvious from $E$-infinity theory and via direct computer simulations. The 120-cell and 600-cell are dual to each other (Fig. 2) and were used by the author in fuzzy form to model a fuzzy Kähler and thus $E$-infinity [7]. Two 600-cell could be made to form one Gosset figure in 8 dimensions.

In this connection one must mention the remarkable crisp 10-dimensional polytopes found by Prof. Ji-Huan He in Shanghai to have 8832 times two elements [8]. A low energy reduction by a 128 component times two spinor leads to a standard model with 69 elementary particles

$$N(\text{SM}) = \frac{8832}{128} = 69$$

which is the integer approximation for the author’s exact result $\langle N(\text{SM}) \rangle = \bar{z}_0/2$.

5. Conclusion

$E$-infinity which is the basis of the newly developed quantum golden field theory rests on transfinite scaling transformation. The transfinite version of $E_8$ exceptional lie group can serve as the central piece of this theory. The power of this theory is manifest not only in giving general explanations to an otherwise complex subject, but also in giving simple quantitative exact solutions to hopelessly nonlinear computation. For instance the quantum gravity inverse coupling of total unification of all fundamental forces may be found as

$$\bar{a}_{GQ} = \left[ \frac{\bar{a}_L}{\bar{a}_{GQ}} \right] = 10 + \left[ \frac{1}{1/2} \right] (\bar{a}_{GQ}) (\varphi^3) = \left[ \frac{42 + 2k}{26 + k} \right].$$

Similarly quarks confinement may be proven [11]. In a manner of speaking and rephrasing the famous words of the pop heros of my generation “All you need is the transfinite $E_8$ and the transfinite $E_8$ is all you need.”

References


